Memo: On computation of the atmosphere excitation function of the Earth rotation

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According to Gross (2007), the excitation function is

$$\begin{aligned} \chi_1 &= \alpha_p \, \Delta I_{13}(t) + \beta_p \, h_1(t) \\ \chi_2 &= \alpha_p \, \Delta I_{23}(t) + \beta_p \, h_2(t) \\ \chi_3 &= \alpha_u \, \Delta I_{33}(t) + \beta_u \, h_3(t) \ , \end{aligned} \tag{1}$$

where $\alpha_p, \alpha_u, \beta_p, \beta_u$ are the following constants:

$$\begin{aligned} \alpha_p &= \frac{\Omega(1 + k'_2 + \Delta k'_a)}{(C_t - (A_t + B_t)/2 + (A_m + B_m)/2 + \varepsilon_c A_c) \, \sigma_{cw}} \\ \alpha_u &= \frac{k_r \, (1 + \alpha_3(k'_2 + \Delta k'_a))}{C_m} \\ \beta_p &= \frac{1}{(C_t - (A_t + B_t)/2 + (A_m + B_m)/2 + \varepsilon_c A_c) \, \sigma_{cw}} \\ \beta_u &= \frac{k_r}{\Omega \, C_m} \ . \end{aligned} \tag{2}$$

Components of momentum of inertia are expressed via these integrals:

$$\begin{aligned} \Delta I_{13}(t) &= -\int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \int\limits_{h_b}^{h_t} (h + R_{\rm ell}(\varphi))^4 \ \rho(h) \cos^2\varphi \, \sin\varphi \, \cos\lambda \, d\varphi \, d\lambda \, dh \\ \Delta I_{23}(t) &= -\int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \int\limits_{h_b}^{h_t} (h + R_{\rm ell}(\varphi))^4 \ \rho(h) \cos^2\varphi \, \sin\varphi \, \sin\lambda \, d\varphi \, d\lambda \, dh \\ \Delta I_{33}(t) &= \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \int\limits_{h_b}^{h_t} (h + R_{\rm ell}(\varphi))^4 \ \rho(h) \cos^3\varphi \, d\varphi \, d\lambda \, dh \end{aligned} \tag{3}$$

and

$$
\begin{aligned}
h_1(t) &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_{h_b}^{h_t} (h+R_{\rm ell}(\varphi))^3\ \rho(\varphi,\lambda,h)\Big(-u(\varphi,\lambda,h)\,\sin\varphi\cos\lambda\ +\ v(\varphi,\lambda,h)\,\sin\lambda\Big)\cos\varphi\,d\varphi\,d\lambda\,dh \\
h_2(t) &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_{h_b}^{h_t} (h+R_{\rm ell}(\varphi))^3\ \rho(\varphi,\lambda,h)\Big(-u(\varphi,\lambda,h)\,\sin\varphi\sin\lambda\ -\ v(\varphi,\lambda,h)\,\cos\lambda\Big)\cos\varphi\,d\varphi\,d\lambda\,dh \\
h_3(t) &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2}\int_{h_b}^{h_t} (h+R_{\rm ell}(\varphi))^3\ \rho(\varphi,\lambda,h)\,u(\varphi,\lambda,h)\,\cos^2\varphi d\varphi\,d\lambda\,dh\ ,
\end{aligned}
\tag{4}
$$

where $R_{\rm ell}(\varphi,\lambda)$ is the distance from the geocenter to the point on the reference ellipsoid at geodetic latitude φ and longitude λ. h is the height above the reference ellipsoid.

Let us introduce integrals:

$$
\begin{aligned}
U(\varphi,\lambda) &= \int_{h_b}^{h_t} \rho(\varphi,\lambda,h)\ \frac{(h+R_{\rm ell}(\varphi))^3}{R_\oplus^3}\,u(\varphi,\lambda,h)\,dh \\
V(\varphi,\lambda) &= \int_{h_b}^{h_t} \rho(\varphi,\lambda,h)\ \frac{(h+R_{\rm ell}(\varphi))^3}{R_\oplus^3}\,v(\varphi,\lambda,h)\,dh \quad . \\
W(\varphi,\lambda) &= \int_{h_b}^{h_t} \rho(\varphi,\lambda,h)\ \frac{(h+R_{\rm ell}(\varphi))^4}{R_\oplus^4}\,dh
\end{aligned}
\tag{5}
$$

Considering that $h \ll R_{\rm ell}$, we replace $h+R_{\rm ell}$ with $R_\oplus\left(1+\dfrac{h+R_{\rm ell}-R_\oplus}{R_\oplus}\right)$, and expand $(h+R_{\rm ell})^a$ in Taylor series retaining the linear and quadratic terms:

$$
\begin{aligned}
U(\varphi,\lambda) &= \int_{h_b}^{h_t} \rho(\varphi,\lambda,h)\ \left(1+3\frac{h+R_{\rm ell}(\varphi)-R_\oplus}{R_\oplus}+3\left(\frac{h+R_{\rm ell}(\varphi)-R_\oplus}{R_\oplus}\right)^2\right)\ v(\varphi,\lambda,h)\,dh \\
V(\varphi,\lambda) &= \int_{h_b}^{h_t} \rho(\varphi,\lambda,h)\ \left(1+3\frac{h+R_{\rm ell}(\varphi)-R_\oplus}{R_\oplus}+3\left(\frac{h+R_{\rm ell}(\varphi)-R_\oplus}{R_\oplus}\right)^2\right)\ u(\varphi,\lambda,h)\,dh\ . \\
W(\varphi,\lambda) &= \int_{h_b}^{h_t} \rho(\varphi,\lambda,h)\ \left(1+4\frac{h+R_{\rm ell}(\varphi)-R_\oplus}{R_\oplus}+6\left(\frac{h+R_{\rm ell}(\varphi)-R_\oplus}{R_\oplus}\right)^2\right)\ dh
\end{aligned}
\tag{6}
$$

The difference between the distance from the point at the ellipsoid with coordinate φ and the equatorial radius is

$$
R_{\rm ell}(\varphi) - R_\oplus = R_\oplus\left(\sqrt{1-e_\oplus^2\sin^2\varphi}-1\right), \tag{7}
$$

where $e_\oplus$ is the eccentricity of the Earth's figure. Integration is performed from the h_b that corresponds to the Earth's surface according to the digital elevation map through the nominal top of the atmosphere: fixed height 80,000 km. Height h is counted from the reference ellipsoid.

Density of moist air obeys the law of state:

$$\rho = \Big((P - P_w)\, M_d + P_w\, M_w \Big) \, \frac{Z^{-1}(P, P_w, T)}{R\, T} \tag{8}$$

where P is the total atmospheric pressure, P_w is the partial pressure of water vapor, T is the absolute temperature, M_d and M_w are molar mass of dry air and water vapor respectively, R is the universal gas constant, and Z^{-1} is the inverse compressibility of moist air.

Then integrals 3–4 can be re-written as

$$\begin{array}{lcl} \Delta I_{13}(t) & = & -R_\oplus^4 \displaystyle\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} W(\varphi,\lambda)\, \cos^2\varphi\, \sin\varphi\, \cos\lambda\, d\varphi\, d\lambda \\ \Delta I_{23}(t) & = & -R_\oplus^4 \displaystyle\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} W(\varphi,\lambda)\, \cos^2\varphi\, \sin\varphi\, \sin\lambda\, d\varphi\, d\lambda \\ \Delta I_{33}(t) & = & R_\oplus^4 \displaystyle\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} W(\varphi,\lambda)\, \cos^3\varphi\, d\varphi\, d\lambda \end{array} \tag{9}$$

and

$$\begin{array}{lcl} h_1(t) & = & R_\oplus^3 \displaystyle\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \Big(-U(\varphi,\lambda)\, \sin\varphi \cos\lambda \, + \, V(\varphi,\lambda)\, \sin\lambda \Big) \cos\varphi\, d\varphi\, d\lambda \\ h_2(t) & = & R_\oplus^3 \displaystyle\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \Big(-U(\varphi,\lambda)\, \sin\varphi \sin\lambda \, - \, V(\varphi,\lambda)\, \cos\lambda \Big) \cos\varphi\, d\varphi\, d\lambda \\ h_3(t) & = & R_\oplus^3 \displaystyle\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} U(\varphi,\lambda)\, \cos^2\varphi\, d\varphi\, d\lambda \end{array} \tag{10}$$

Geodetic parameters of the Earth (Gross, 2007):

G	=	$6.67259 \cdot 10^{-11}$	$m^3/kg\ s^2$
M_{atm}	=	$5.1441 \cdot 10^{18}$	kg
M_{ocn}	=	$1.4 \cdot 10^{21}$	kg

Whole Earth (observed)

$R_\oplus$	=	6378137.0	m
$e^2_\oplus$	=	0.0033528132	
$\Omega_\oplus$	=	$7.2921151467 \cdot 10^{-5}$	rad/s
σ_{cw}	=	$1.67485 \cdot 10^{-7}$	rad/s
$M_\oplus$	=	$5.9737 \cdot 10^{24}$	kg
C_t	=	$8.0365 \cdot 10^{37}$	kg m^2
B_t	=	$8.0103 \cdot 10^{37}$	kg m^2
A_t	=	$8.0101 \cdot 10^{37}$	kg m^2
$C_t - A_t$	=	$2.6398 \cdot 10^{35}$	kg m^2
$C_t - B_t$	=	$2.6221 \cdot 10^{35}$	kg m^2
$B_t - A_t$	=	$1.765 \cdot 10^{33}$	kg m^2
a	=	6371000	m
k_2	=	0.298	
$k_{ocn,w}$	=	0.047715	
$k_{ocn,s}$	=	0.043228	
k_r	=	0.997191	
k'_2	=	-0.305	
$\Delta k'_a$	=	-0.011 + i 0.003	
α_3	=	0.792	

Crust and mantle (PREM)

ε_a	=	$3.334 \cdot 10^{-3}$	
$M_m =$	=	$4.0337 \cdot 10^{24}$	kg
$C_m =$	=	$7.1236 \cdot 10^{37}$	kg m^2
$A_m =$	=	$7.0999 \cdot 10^{37}$	kg m^2

Core (PREM)

ε_c	=	$2.546 \cdot 10^{-3}$	
M_c	=	$1.9395 \cdot 10^{24}$	kg
$C_c =$	=	$9.1401 \cdot 10^{36}$	kg m^2
$A_c =$	=	$9.1168 \cdot 10^{36}$	kg m^2

Finally, can write the excitation function in the form

$$\begin{aligned}
\chi_1 &= \alpha_p \, \Delta I_{13}(t) + \beta_p \, h_1(t) \\
\chi_2 &= \alpha_p \, \Delta I_{23}(t) + \beta_p \, h_2(t) \\
\chi_3 &= \alpha_u \, \Delta I_{33}(t) + \beta_u \, h_3(t) \qquad (11)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_p &= \frac{\Omega(1 + k_2' + \Delta k_a')}{(C_t - (A_t + B_t)/2 + (A_m + B_m)/2 + \varepsilon_c A_c)\,\sigma_{cw}} = 4.17767 \cdot 10^{-36} \quad \mathrm{m^2/(s \cdot kg)} \\
\alpha_u &= \frac{k_r\,(1 + \alpha_3(k_2' + \Delta k_a'))}{C_m} = 1.04950 \cdot 10^{-38} \quad \mathrm{m^2/(s \cdot kg)} \\
\beta_p &= \frac{1}{(C_t - (A_t + B_t)/2 + (A_m + B_m)/2 + \varepsilon_c A_c)\,\sigma_{cw}} = 8.37576 \cdot 10^{-32} \quad \mathrm{m^2/kg} \\
\beta_u &= \frac{k_r}{\Omega\, C_m} = 1.91966 \cdot 10^{-34} \quad \mathrm{m^2/kg}
\end{aligned}$$

References

Gross, R. S., Earth Rotation Variations – Long Period, in Physical Geodesy, edited by T. A. Herring, Treatise on Geophysics, Vol. 11, Elsevier, Amsterdam, in press, 2007.